Solution 4 :

1°/ $\lim\limits_{x\to 0}\sqrt{\dfrac{\sin x}{x}}=\sqrt{1}=1$.

2°/ $\lim\limits_{x\to 0}\dfrac{\sin x+\sin 2x}{\sin x-\sin 2x}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{x}+\dfrac{\sin 2x}{x}}{\dfrac{\sin x}{x}-\dfrac{\sin 2x}{x}}=\dfrac{1+2}{1-2}=-3$

(car $\lim\limits_{x\to 0}\dfrac{\sin ax}{bx}=\dfrac{a}{b}$).

3°/ on a : $5\cos^2 x+\sin^2 x-4\cos x=5\cos^2 x+1-\cos^2 x-4\cos x$

$=4\cos^2 x-4\cos x+1=(2\cos x-1)^2$ donc :

$\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sin x}{5\cos^2 x+\sin^2 x-4\cos x}=\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sin x}{(2\cos x-1)^2}=+\infty$ car $\sin\dfrac{\pi}{3}>0$

et $\lim\limits_{x\to\frac{\pi}{3}}(2\cos x-1)^2=0^+$.

4°/ On a, d'une part $\forall x\in\mathbf{R}, -1\le\sin x\le 1 \Rightarrow 1\le 2+\sin x\le 3 \Rightarrow$

$\dfrac{1}{3}\le\dfrac{1}{2+\sin x}\le 1$ **(1)**

et d'autre part : $-1\le-\sin^2 x\le 0\Rightarrow x-1\le x-\sin^2 x\le x$ **(2)**.

En multipliant les 2 encadrements (1) et (2) on obtient : $\forall x>1$

$\dfrac{x-1}{3}\le\dfrac{x-\sin^2 x}{2+\sin x}\le x \Rightarrow \forall x>1,\ \dfrac{x-1}{3}\le\dfrac{x-\sin^2 x}{2+\sin x}$ or

$\lim\limits_{x\to+\infty}\dfrac{x-1}{3}=+\infty$ donc $\lim\limits_{x\to+\infty}\dfrac{x-\sin^2 x}{2+\sin x}=+\infty$.